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ON THE STABILITY OF APPROXIMATION OPERATORS IN PROBLEMS OF STRUCTURAL DYNAMICS

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Abstract—Three direct integration schemes for the matrix equations of motion of structural dynamics—the Newmark generalized acceleration operator, the Wilson averaging variant of the linear acceleration operator and an averaging method based on a variational principle derived by Gurtin—are investigated for stability and approximation viscosity. Using established techniques developed by J. von Neumann and Lax and Richtmyer, the latter two approximation operators are found to be unconditionally stable. In addition, the constant average acceleration version of the Newmark method is found to be unconditionally stable and to possess no attenuation due to approximation viscosity. Truncation error due to the low-pass filtering characteristics of spatially discretized systems is highly damped by the Wilson averaging and Gurtin averaging operators; all three operators exhibit error in the period of the response which is a function of time step size.

1. INTRODUCTION

AS THE tools of analysis become more versatile, dynamic design requirements will be treated in much the same manner as static requirements—using dynamic failure data, accumulative damage concepts and safety margins. A particular analytical tool which has been used extensively in static stress analysis and which offers tangible promise for evaluating the dynamic response of structures is the finite element method, a direct variational procedure based on Ritz spatial approximation [1, 2].

The success of finite element methods when applied to the forced dynamic response of structures has, for the most part, been more apparent than real. Most of the step-forward integration schemes in use today had their origins in forced response calculations for modally decomposed systems where the primary emphasis was on the lowest natural modes of the structure. As a result, stability problems were of minor concern; in general, these integration schemes are conditionally stable for a step size small compared to the natural period of a one-degree-of-freedom system. In recent years, however, direct integration of the equations of motion has become the more popular approach [3, 4]. In this case, all of the natural modes implicitly influence the integration procedure at each time step, greatly complicating considerations of convergence and stability. Practitioners of the art who are able to judiciously select time step size in order to achieve the elusive objectives of solution stability and reasonable computation time are scarce. Occasional spurious results often defy careful post-computation analysis.

It would seem clear, then, that, if finite element techniques are to achieve the same goals in dynamic stress analysis that have been achieved in static stress analysis, an improved rationale for treating the temporal variation should be sought.

2. PRELIMINARIES. THE APPROXIMATION OPERATOR

In order to discuss the finite element method in this context, it is appropriate to recall some useful definitions and notations. For the most part, the treatment outlined here follows closely that developed by O'Brien *et al.* [5], who discuss the stability analysis originated by J. von Neumann, and the classical work of Lax and Richtmyer [6]. The difference between these two approaches allegedly concerns the source of error in the approximate solution and the tendency for these errors to grow without bound as the solution is continued.

There are three distinct solutions to the initial-boundary-value problem: (1) the exact solution of the governing partial differential equation; (2) the exact solution to the approximate equations obtained through spatial and temporal discretization; and (3) the numerical solution of the approximate equations, considering finite-precision arithmetic. Truncation error is the measure of the difference between the first two solutions and round-off error is the measure of the last solutions. The methods developed by von Neumann and those used by Lax and Richtmyer are equally applicable to both types of error, however, since both are eventually concerned with the spectral decomposition of the approximation operator. It would seem that the source of the error, whether it be from initial or boundary data that the discretized system cannot describe or from finite-precision arithmetic by the computer, is not as important as the spectral character of the approximation operator.

In the remainder of this work, the discussion will be limited to the displacement formulation of the finite element method as applied to the linear initial-boundary-value problem of structural dynamics; i.e. the nodal point displacements, velocities and accelerations at time t_n are defined in a Hilbert space H and the nodal point values at time t_{n+1} are related to those at t_n through a linear transformation. This linear transformation, generally a function of the time step size, $\Delta t = t_{n+1} - t_n$, and the stiffness and inertial properties of the structure, will be referred to as the approximation operator. (In [6] this operator is denoted the amplification matrix.) It should be noted that the proper formulation of this problem requires only the one-step transformation described above; however, the linearity of the operator implies that equivalent (in terms of spectral representation) multi-level formulas can be deduced.

To be more precise, let $\mathbf{u}(t)$ be the vector in H representing the nodal point displacements, velocities and accelerations. Through some procedure, such as applying a restricted first variation to Hamilton's principle or through the Gurtin variational principle [7], the matrix equation

$$\mathscr{K}_1 \mathbf{u}(t_{n+1}) = \mathbf{F} + \mathscr{K}_0 \mathbf{u}(t_n) \tag{2.1}$$

is found. The matrices $\mathscr{K}_0$ and $\mathscr{K}_1$ and the vector $\mathbf{F}$ are, in general, functions of Δt and the physical parameters of the structure. If the matrix $\mathscr{K}_1$ can be written in triangular form, the formula (2.1) is defined to be explicit; otherwise, the formula is implicit. The approximation operator is, of course,

$$\mathscr{A} = \mathscr{K}_1^{-1} \mathscr{K}_0, \tag{2.2}$$

assuming that the inverse of the matrix $\mathscr{K}_1$ exists. Then,

$$\mathbf{u}(t_{n+1}) = \mathbf{G} + \mathscr{A}\mathbf{u}(t_n), \tag{2.3}$$

where

$$\mathbf{G} = \mathscr{K}_1^{-1}\mathbf{F}. \tag{2.4}$$

If the approximation operator is consistent,† then

$$\lim_{\Delta t \to 0} \left\{ \frac{\mathbf{u}(t_{n+1}) - \mathbf{G} - \mathscr{A}\mathbf{u}(t_n)}{\Delta t} \right\}$$

provides a consistent approximation to the time derivatives of $\mathbf{u}(t)$. The Lax Equivalence Theorem then states that a consistent approximation operator for a properly posed initial-boundary-value problem is convergent if and only if the approximation operator is stable. Convergence in this sense is defined to be convergence in the norm of H. Stability is defined in the usual way; i.e. the approximation operator is uniformly bounded. The question is thus resolved to be whether or not, for a given time step size Δt, the approximation operator has bounded spectral radius.

In the sections to follow, a number of popular approximation operators are investigated in light of the Lax Equivalence Theorem. It should be pointed out at this time that at least two approximation operators—the standard central difference formula and the Houbolt backward difference operator [8]—have been investigated for stability by using the von Neumann procedure to estimate the spectral radius. The Houbolt operator was found to be unconditionally stable [9] while the central difference operator is conditionally stable [10]; i.e. for a time step size larger than 4π times the shortest natural period of the structure, the procedure is unstable. These results had been anticipated in earlier work [11].

3. THE NEWMARK GENERALIZED ACCELERATION METHOD

A method for directly integrating the equations of motion of a structural system which has been widely used is the Newmark generalized acceleration method [12]. The nodal point displacements and velocities are approximated by the expressions

$$\mathbf{u}(t_{n+1}) = \mathbf{u}(t_n) + \Delta t\dot{\mathbf{u}}(t_n) + (\tfrac{1}{2} - \beta)(\Delta t)^2\ddot{\mathbf{u}}(t_n) + \beta(\Delta t)^2\ddot{\mathbf{u}}(t_{n+1}) \tag{3.1}$$

and

$$\dot{\mathbf{u}}(t_{n+1}) = \dot{\mathbf{u}}(t_n) + (1 - \gamma)\Delta t\ddot{\mathbf{u}}(t_n) + \gamma\Delta t\ddot{\mathbf{u}}(t_{n+1}) \tag{3.2}$$

where β and γ are the dimensionless parameters of generalized acceleration and $\ddot{\mathbf{u}}(t)$ indicates the nodal point accelerations.

Chan *et al.* [3], have discussed the special case $\beta = \frac{1}{12}$, $\gamma = \frac{1}{2}$, which coincides with a procedure developed by Fox and Goodwin [13]. The constant average acceleration ($\beta = \frac{1}{4}, \gamma = \frac{1}{2}$) and the linear acceleration ($\beta = \frac{1}{6}, \gamma = \frac{1}{2}$) methods are also special cases.

For a one-degree-of-freedom system, stability can be investigated by using the Lax–Richtmyer approach. For the case where $\gamma = \frac{1}{2}$, the nontrivial eigenvalues of the approximation operator are

$$\lambda_{1,2} = \frac{1 + (\beta - 1/2)\xi \pm i[\xi^2(\beta - \frac{1}{4}) + \xi - \eta^2/4]^{\frac{1}{2}}}{1 + \eta/2 + \beta\xi}, \tag{3.3}$$

† The term consistent, in this sense, can be interpreted to mean that the difference between a power series expansion of the time derivatives of $\mathbf{u}(t)$ and the approximation operator contains only quantities that vanish as $\Delta t \to 0$.

where $\eta = C\Delta t/M$ and $\xi = (\Delta t)^2 K/M$; K, C and M are the stiffness, damping and mass of the system.

In order for the solution to be oscillatory for the case of zero damping,

$$\beta \geq \tfrac{1}{4}. \tag{3.4}$$

This same result can be obtained by taking the limit of (3.3) as the time step grows large and insisting that the absolute value of the eigenvalues be bounded from above by unity.

Similar results can be obtained for a multi-degree-of-freedom system by using the von Neumann method with equation (17) of [3]:

$$\left[[M]+\frac{\Delta t}{2}[C]+\beta(\Delta t)^2[K]\right]\{u(t_{n+1})\} = (\Delta t)^2\{\beta\{F(t_{n+1})\}+(1-2\beta)\{F(t_n)\}+\beta\{F(t_{n-1})\}\}$$
$$+2[[M]-(\Delta t)^2(\tfrac{1}{2}-\beta)[K]]\{u(t_n)\}$$
$$-\left[[M]-\frac{\Delta t}{2}[C]+\beta(\Delta t)^2[K]\right]\{u(t_{n-1})\}. \tag{3.5}$$

To investigate stability for an undamped system, the error in the numerical solution at time $t = t_n = n\Delta t$ is assumed to be given by

$$\{\varepsilon(t_n)\} = e^{\alpha n\Delta t}\{d\}, \tag{3.6}$$

where $\{d\}$ is a vector of arbitrary nodal point errors. Defining the characteristic value

$$\lambda = e^{\alpha\Delta t} \tag{3.7}$$

and noting that the error must satisfy the homogeneous form of equation (3.5), then

$$([M]^{-1}[K]-\gamma[I])\{d\} = 0, \tag{3.8}$$

where

$$\gamma = -\frac{(\lambda-1)^2}{\beta(\Delta t)^2[\lambda^2+(1/\beta-2)\lambda+1]}. \tag{3.9}$$

But (3.8) can be recognized as the characteristic equation for the natural frequencies of the finite element system. Then, since all these frequencies are real and positive (or zero for rigid body modes) for the finite element formulation, the eigenvalues of the approximation operator can be determined, in pairs, as functions of the time step size Δt and each of the system natural frequencies:

$$\lambda_{1,2} = \frac{1+(\beta-\frac{1}{2})\omega^2(\Delta t)^2 \pm i[(\beta-\frac{1}{4})\omega^4(\Delta t)^4+\omega^2(\Delta t)^2]^{\frac{1}{2}}}{1+\beta\omega^2(\Delta t)^2}. \tag{3.10}$$

The condition that the spectral radius be bounded dictates that the Newmark method is unconditionally stable provided that $\beta \geq \frac{1}{4}$, as before. This result was obtained by a different procedure in [12] for a one-degree-of-freedom system.

The eigenvalues of the approximation operator can be written in polar form for $\beta = \frac{1}{4}$ as

$$\lambda_{1,2} = Re^{\pm i\theta}, \tag{3.11}$$

where

$$R = 1, \theta = \tan^{-1}\left(\frac{\omega\Delta t}{1-\frac{1}{4}\omega^2(\Delta t)^2}\right), \tag{3.12}$$

which shows that there is no artificial or inherent damping in the approximation operator.

In an effort to verify the results obtained in this and subsequent sections, the various approximation operators under consideration were used to solve a simple one-degree-of-freedom problem—that of a linear oscillator subjected to a step force in time. The exact solution is

$$\mathbf{u}(t) = \frac{F_0}{\Omega^2}[1-\cos\Omega t], \tag{3.13}$$

where u is the displacement, F_0 is the applied force, Ω is the natural frequency for a unit mass and the initial conditions have been chosen to be quiescent.

A comparison between the exact solution and the constant average acceleration approximation operator is shown in Figs. 1–3 for time step sizes $\Delta t = 0{\cdot}2$, $0{\cdot}5$ and $1{\cdot}0$. The agreement between maximum and minimum values generated by the numerical solution and the exact solution verify the lack of damping in this operator; there is an error in the vibratory period, however, as indicated by (3.12b). To illustrate this, the exact solution to the difference equation is

$$u(t_n) = AR^n e^{in\theta} + BR^n e^{-in\theta} \tag{3.14}$$

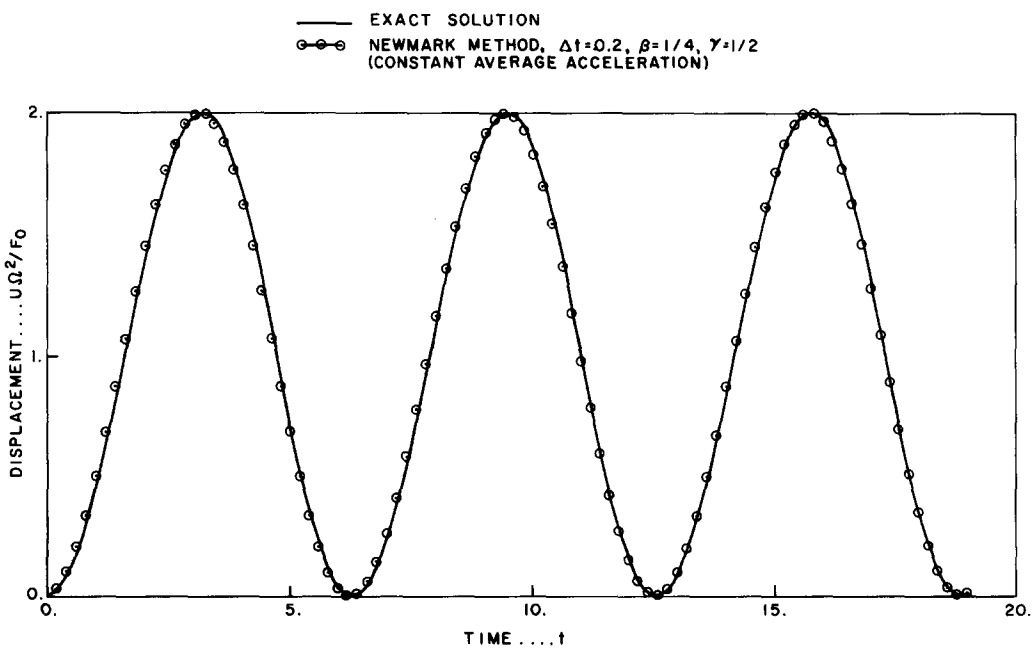

FIG. 1. Comparison of Newmark operator ($\beta = \frac{1}{4}$) to exact solution for $\Delta t = 0{\cdot}2$.

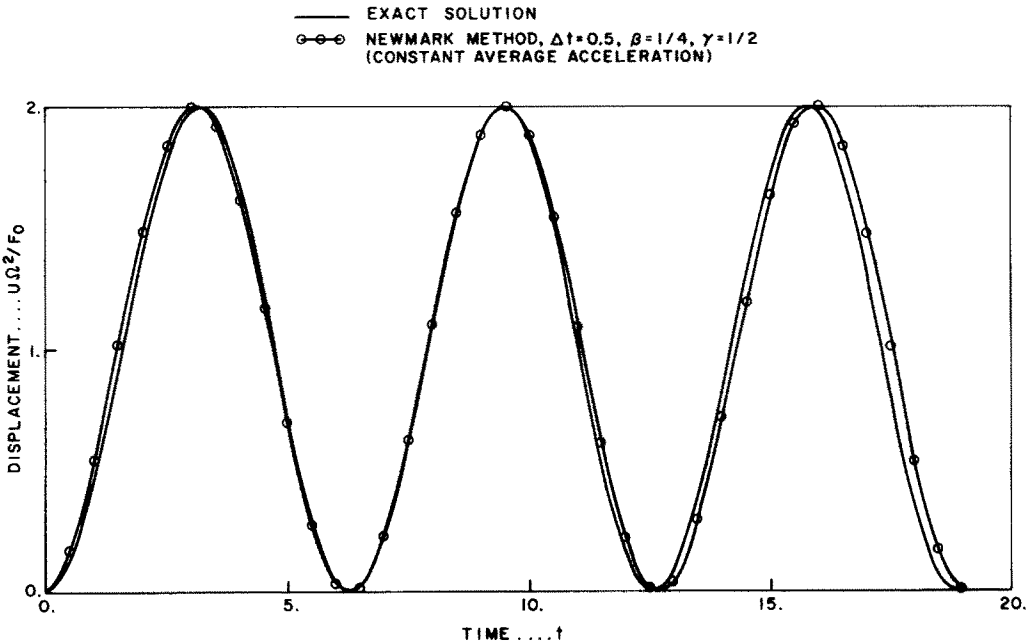

FIG. 2. Comparison of Newmark operator ($\beta = \frac{1}{4}$) to exact solution for $\Delta t = 0{\cdot}5$.

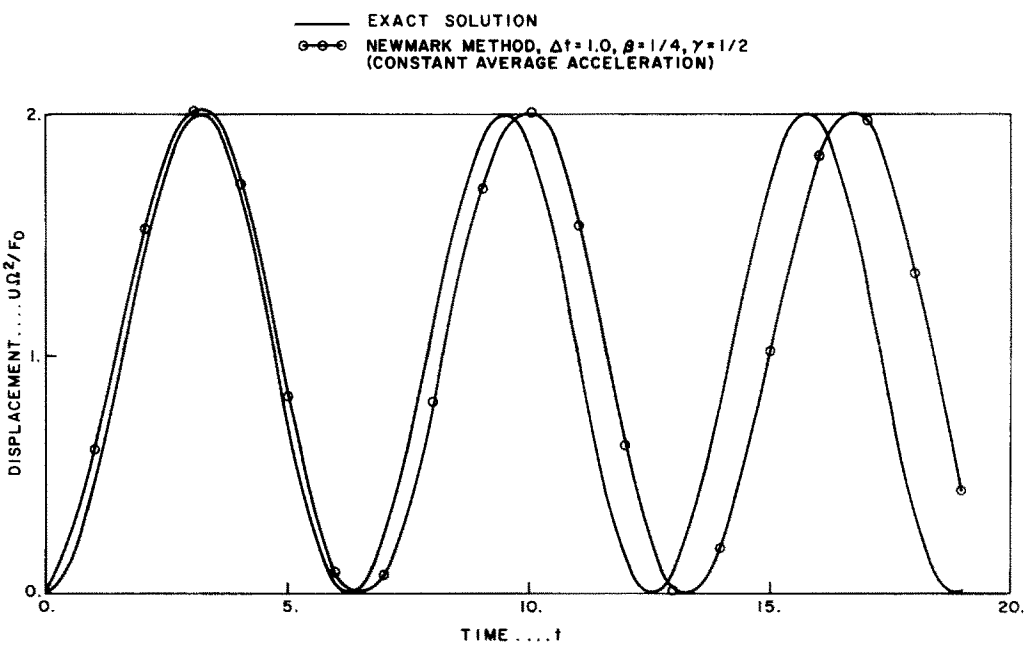

FIG. 3. Comparison of Newmark operator ($\beta = \frac{1}{4}$) to exact solution for $\Delta t = 1{\cdot}0$.

or, using (3.12a) and noting that

$$n = \frac{t_n}{\Delta t},$$

$$u(t_n) = A' \cos\left(\frac{\theta t_n}{\Delta t}\right) + B' \sin\left(\frac{\theta t_n}{\Delta t}\right), \tag{3.15}$$

where the constants A' and B' are determined from the initial conditions. The maxima and minima of equation (3.15) will occur for

$$\frac{\theta t_n}{\Delta t} = m\pi, \qquad m = 0, 1, 2, \ldots. \tag{3.16}$$

or at

$$t_n = \frac{\Delta t(m\pi)}{\theta}. \tag{3.17}$$

Calculating θ from (3.12b) for $\Delta t = 1$, the maximum which occurs at 3π in the exact solution is shifted to $t \doteq 10{\cdot}15$ in the exact difference solution while the minimum at 4π is shifted to $t = 13{\cdot}54$. The numerical solution was not obtained at these points but these values are in essential agreement with the faired curve of Fig. 3.

4. THE WILSON AVERAGING OPERATOR

A modification of the linear acceleration method has been successfully applied to a large class of plane [14] and axisymmetric [15] wave propagation problems. The matrix equation of motion for the system is written at time $t = t_{n+1}$; then, the nodal point accelerations are assumed to vary linearly in the interval (t_n, t_{n+1}). This implies that the nodal point displacement vector is expanded as a cubic with coefficient vectors defined in terms of initial values (at $t = t_n$) of the displacement, velocity, and acceleration and the unknown displacements at t_{n+1}. Then

$$\mathbf{u}(\tau) = \mathbf{B}_0 + (\tau - t_n)\mathbf{B}_1 + \tfrac{1}{2}(\tau - t_n)^2\mathbf{B}_2 + \tfrac{1}{6}(\tau - t_n)^3\mathbf{B}_3 \tag{4.1}$$

or

$$\mathbf{u}(\tau) = \frac{(\tau - t_n)^3}{(\Delta t)^3}\mathbf{u}(t_{n+1}) + \left[\frac{(\Delta t)^3 - (\tau - t_n)^3}{(\Delta t)^3}\right]\mathbf{u}(t_n)$$
$$+ \left[\frac{(\Delta t)^2(\tau - t_n) - (\tau - t_n)^3}{(\Delta t)^2}\right]\dot{\mathbf{u}}(t_n) + \frac{1}{2}\left[\frac{\Delta t(\tau - t_n)^2 - (\tau - t_n)^3}{\Delta t}\right]\ddot{\mathbf{u}}(t_n), \tag{4.2}$$

where $t_n \leq \tau \leq t_{n+1}$.

Taking appropriate time derivatives and evaluating the results at time $\tau = t_{n+1}$ gives expressions for the nodal point velocities and accelerations

$$\dot{\mathbf{u}}(t_{n+1}) = \frac{3}{\Delta t}\mathbf{u}(t_{n+1}) - \frac{3}{\Delta t}\mathbf{u}(t_n) - 2\dot{\mathbf{u}}(t_n) - \frac{\Delta t}{2}\ddot{\mathbf{u}}(t_n) \tag{4.3}$$

$$\ddot{\mathbf{u}}(t_{n+1}) = \frac{6}{(\Delta t)^2}\mathbf{u}(t_{n+1}) - \frac{6}{(\Delta t)^2}\mathbf{u}(t_n) - \frac{6}{\Delta t}\dot{\mathbf{u}}(t_n) - 2\ddot{\mathbf{u}}(t_n). \tag{4.4}$$

If (4.4) is solved for $\mathbf{u}(t_{n+1})$ and substituted into (4.3), the resulting expressions are identical to the Newmark operator with $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$. Since the linear acceleration method can be shown to be conditionally stable, the operator was modified to reflect midpoint values at

$$t_m = \tfrac{1}{2}(t_n + t_{n+1}). \tag{4.5}$$

From (4.2)

$$\mathbf{u}(t_m) = \frac{1}{8}\mathbf{u}(t_{n+1}) + \frac{7}{8}\mathbf{u}(t_n) + \frac{3}{8}\Delta t\dot{\mathbf{u}}(t_n) + \frac{1}{16}(\Delta t)^2\ddot{\mathbf{u}}(t_n), \tag{4.6}$$

$$\dot{\mathbf{u}}(t_m) = \frac{3}{4(\Delta t)}\mathbf{u}(t_{n+1}) - \frac{3}{4(\Delta t)}\mathbf{u}(t_n) + \frac{1}{4}\dot{\mathbf{u}}(t_n) + \frac{1}{8}\Delta t\ddot{\mathbf{u}}(t_n), \tag{4.7}$$

and

$$\ddot{\mathbf{u}}(t_m) = \frac{3}{(\Delta t)^2}\mathbf{u}(t_{n+1}) - \frac{3}{(\Delta t)^2}\mathbf{u}(t_n) - \frac{3}{\Delta t}\dot{\mathbf{u}}(t_n) - \frac{1}{2}\ddot{\mathbf{u}}(t_n). \tag{4.8}$$

Equation (4.6) can be solved for $\mathbf{u}(t_{n+1})$ and these nodal point displacements eliminated from (4.7) and (4.8). Then

$$\dot{\mathbf{u}}(t_m) = \frac{6}{\Delta t}\mathbf{u}(t_m) - \frac{6}{\Delta t}\mathbf{u}(t_n) - 2\dot{\mathbf{u}}(t_n) - \frac{1}{4}\Delta t\ddot{\mathbf{u}}(t_n) \tag{4.9}$$

and

$$\ddot{\mathbf{u}}(t_m) = \frac{24}{(\Delta t)^2}\mathbf{u}(t_m) - \frac{24}{(\Delta t)^2}\mathbf{u}(t_n) - \frac{12}{\Delta t}\dot{\mathbf{u}}(t_n) - 2\ddot{\mathbf{u}}(t_n). \tag{4.10}$$

The stability of this approach can be investigated using the Lax–Richtmyer procedure. The characteristic equation for the case of zero damping is

$$\lambda^3(1+\tfrac{1}{6}\xi)^3 - \lambda^2(1+\tfrac{1}{6}\xi)^2(\tfrac{5}{2}+\tfrac{5}{48}\xi) + \lambda(1+\tfrac{1}{6}\xi)(2+\tfrac{5}{12}\xi+\tfrac{1}{72}\xi^2) - (\tfrac{1}{2}+\tfrac{3}{16}\xi+\tfrac{1}{48}\xi^2+\tfrac{1}{1728}\xi^3) = 0. \tag{4.11}$$

Following the steps outlined in [9], the spectral radius of the approximation operator can be shown to be bounded by unity, providing a sufficient condition for stability.

The moduli of the eigenvalues of the characteristic equation (4.11) are plotted in Fig. 4 vs. the time parameter ξ. These plots indicate the attenuation in the approximation operator but give no information about period error. The results of applying the Wilson averaging operator to the one-degree-of-freedom problem described in the previous section are shown in Figs. 5–7 (note that the solution increment size is one-half the time step size; this is due to finding the solution at the center of the interval and using these

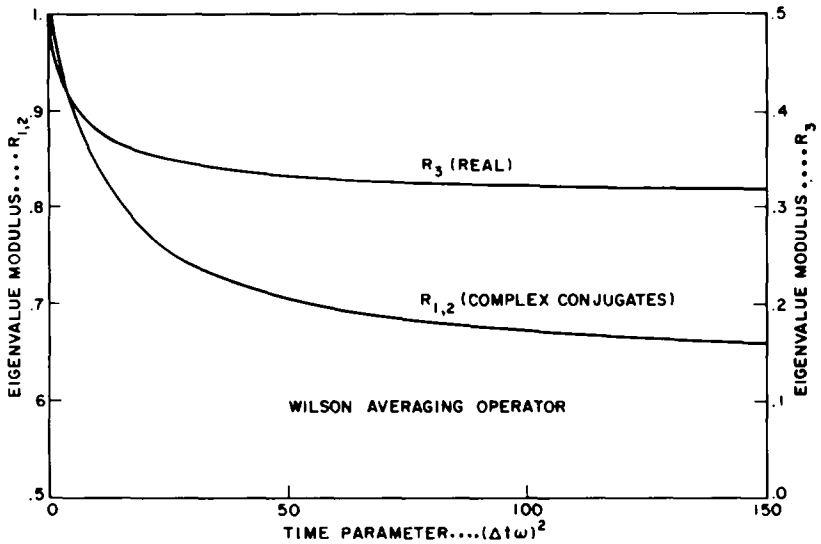

FIG. 4. Eigenvalue moduli for Wilson averaging operator.

values as initial conditions for the next step). The attenuation and period error are strong functions of the time step size as can be seen from these plots.

5. THE GURTIN VARIATIONAL METHOD

Another method, which is based on Ritz approximation in both the space and time variables, has been developed for application to thermoelastic [16] and thermoviscoelastic

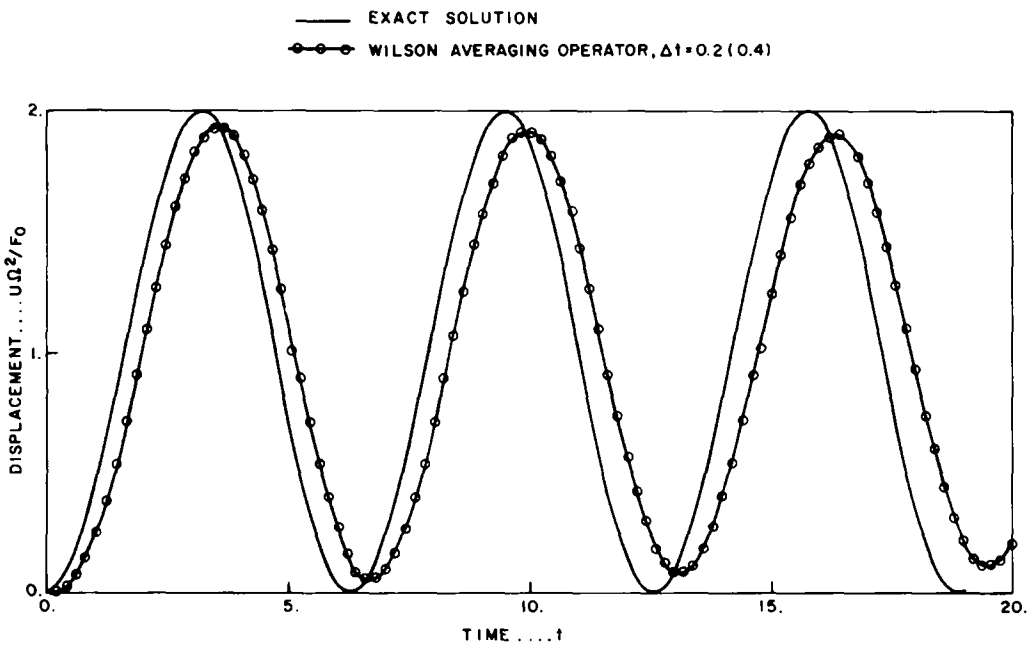

FIG. 5. Comparison of Wilson averaging operator to exact solution for $\Delta t = 0{\cdot}2\ (0{\cdot}4)$.

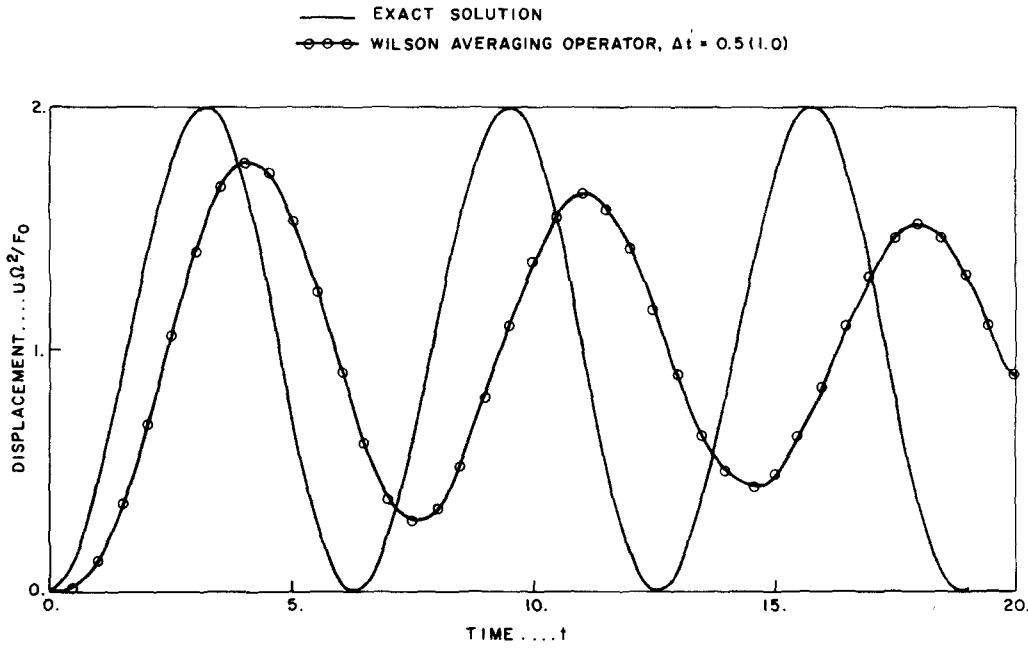

FIG. 6. Comparison of Wilson averaging operator to exact solution for $\Delta = 0{\cdot}5\ (1{\cdot}0)$.

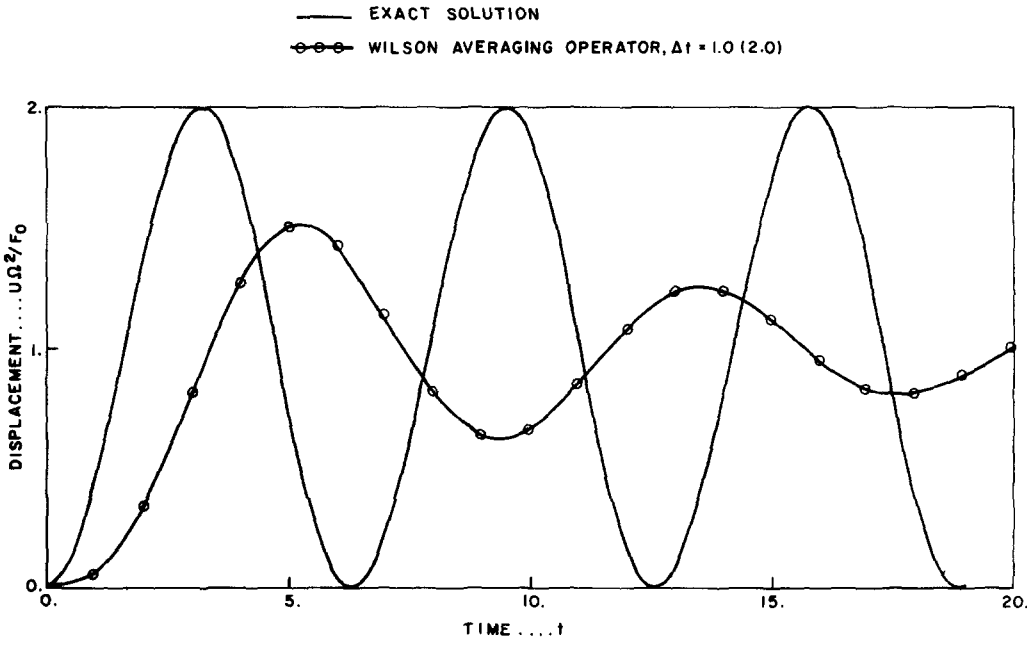

FIG. 7. Comparison of Wilson averaging operator to exact solution for $\Delta t = 1{\cdot}0\ (2{\cdot}0)$.

[17] wave propagation problems. Since the foundations of the method were first discussed by Gurtin in his now classical treatment of elastodynamics through operational variational principles [7], reference will be to the Gurtin method.

For the problems of structural dynamics, the governing equations of motion are given in matrix form by

$$[M]\{\ddot{u}(t)\}+[C]\{\dot{u}(t)\}+[K]\{u(t)\} = \{F(t)\}, \tag{5.1}$$

where $[M]$, $[C]$ and $[K]$ represent the mass, damping and stiffness matrices, respectively, and $\{u(t)\}$, $\{\dot{u}(t)\}$ and $\{\ddot{u}(t)\}$ denote the nodal point displacements, velocities, accelerations and forces, respectively. Taking the Laplace transform (5.1), solving for the transformed nodal point displacements, and inverting gives

$$\begin{aligned}&[M]\{u(t)\}+g'^{*}[C]\{u(t)\}+g^{*}[K]\{u(t)\}\\ &\quad = g^{*}\{F(t)\}+[M]\{u(0)\}+t[M]\{\dot{u}(0)\}+t[C]\{u(0)\},\end{aligned} \tag{5.2}$$

where $\{u(0)\}$ and $\{\dot{u}(0)\}$ are the initial nodal point displacements and velocities; the functions

$$g(t) = t, \qquad g'(t) = 1; \tag{5.3}$$

and the convolution of two functions of time is defined by

$$(f^{*}g)(t) = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau. \tag{5.4}$$

Equation (5.2) is applicable to step-forward integration schemes provided that the time interval $(0, t)$ is interpreted to be (t_n, t_{n+1}). Then, using a quadratic polynomial assumption for the displacement field in time (constant acceleration) similar to that described in [16], the equations of motion can be written

$$\begin{aligned}\left[[M]+\frac{\Delta t}{3}[C]+\frac{(\Delta t)^2}{12}[K]\right]\{u(t_{n+1})\} &= \frac{(\Delta t)^2}{12}\{\{F(t_{n+1})\}+5\{F(t_n)\}+\Delta t\{\dot{F}(t_n)\}\}\\ &\quad+\left[[M]+\frac{\Delta t}{3}[C]-\frac{5}{12}(\Delta t)^2[K]\right]\{u(t_n)\}\\ &\quad+\Delta t\left[[M]-\frac{\Delta t}{6}[C]-\frac{(\Delta t)^2}{12}[K]\right]\{\dot{u}(t_n)\}.\end{aligned} \tag{5.5}$$

The stability of these equations may be investigated in a simple way by developing the approximation matrix for a one-degree-of-freedom system. Then, the non-trivial eigenvalues for this operator (neglecting damping) are

$$\lambda_{1,2} = \frac{1-\frac{1}{3}\xi \pm i[\xi-\frac{13}{144}\xi^2]^{\frac{1}{2}}}{1+\frac{1}{12}\xi}. \tag{5.6}$$

These expressions indicate that the spectral radius for this approximation operator is unbounded for any nonzero value of Δt, implying unconditional instability. To verify this, the numerical and exact solutions for the one-degree-of-freedom system subjected to step load are shown in Fig. 8, indicating error growth which, if the numerical solution were continued, would become infinite.

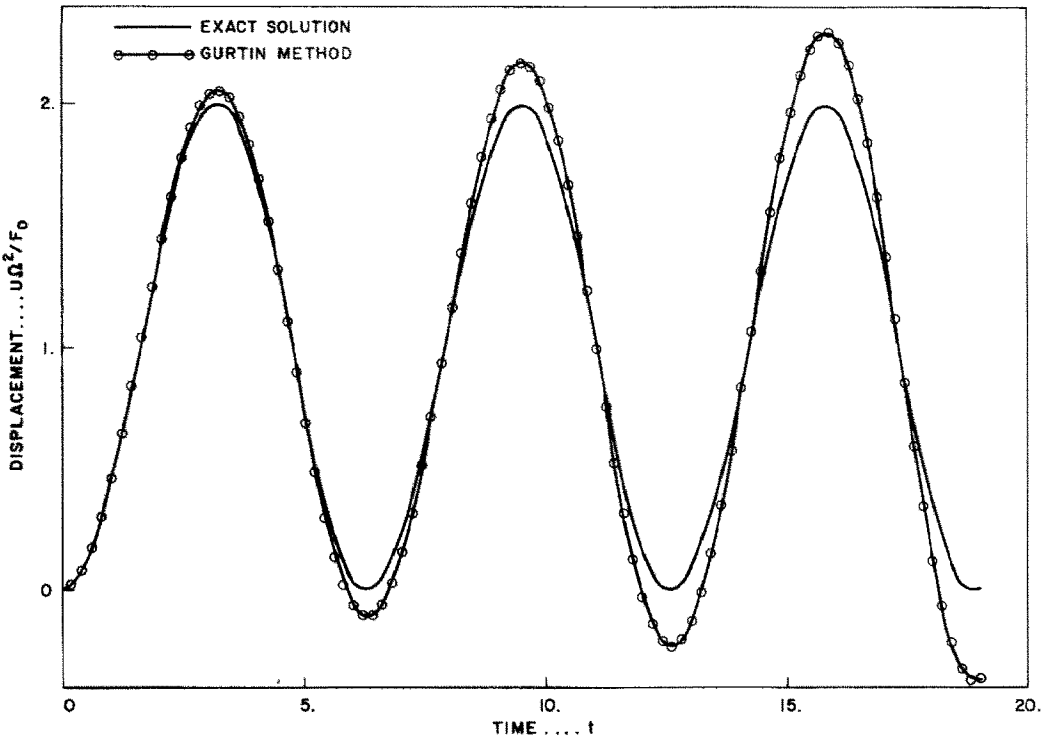

FIG. 8. Comparison of unconditionally unstable operator to exact solution for $\Delta t = 0{\cdot}2$.

A slight alteration of this operator, which was introduced in [17], is similar to the Wilson averaging method in that the numerical solution is sought at middle of the time interval

$$t_a = \tfrac{1}{2}(t_n + t_{n+1}) \tag{5.7}$$

so that

$$\left[[M]+\frac{\Delta t}{3}[C]+\frac{(\Delta t)^2}{12}[K]\right]\{u(t_a)\} = \frac{(\Delta t)^2}{12}\{\{F(t_a)\}+\tfrac{1}{2}\{F(t_n)\}\} + \left[[M]+\frac{\Delta t}{3}[C]-\frac{(\Delta t)^2}{24}[K]\right]\{u(t_n)\} + \tfrac{1}{2}(\Delta t)\left[[M]+\frac{\Delta t}{12}[C]\right]\{u(t_n)\}. \tag{5.8}$$

The stability of these equations can be investigated in two ways, as previously indicated. Using the Lax–Richtmyer procedure for a one-degree-of-freedom system, the non-trivial eigenvalues of the approximation operator are found to be

$$\lambda_{1,2} = \frac{1+\frac{1}{12}\eta-\frac{1}{16}\xi \pm i[\frac{1}{4}\xi-\frac{1}{2304}\xi^2-\frac{1}{16}\eta^2+\frac{1}{96}\eta\xi]^{1/2}}{1+\frac{1}{3}\eta+\frac{1}{12}\xi}. \tag{5.9}$$

When the time step size is allowed to grow large, these eigenvalues become $\lambda = -\frac{1}{2}$ and -1; this indicates that the spectral radius remains finite as the time step grows large, implying that the procedure is unconditionally stable.

These eigenvalues can be written in terms of modulus and phase as

$$\lambda_{1,2} = R\,e^{\pm i\theta}, \tag{5.10}$$

where

$$R = \left[\frac{1+\frac{1}{8}\xi+\frac{1}{6}\eta+\frac{1}{288}\xi^2-\frac{1}{18}\eta^2}{1+\frac{1}{6}\xi+\frac{2}{3}\eta+\frac{1}{144}\xi^2+\frac{1}{9}\eta^2+\frac{1}{18}\eta\xi}\right]^{\frac{1}{2}} \tag{5.11}$$

and

$$\theta = \tan^{-1}\left\{\frac{[\frac{1}{4}\xi-\frac{1}{2304}\xi^2-\frac{1}{16}\eta^2+\frac{1}{96}\eta\xi]^{\frac{1}{2}}}{1+\frac{1}{12}\eta-\frac{1}{16}\xi}\right\}. \tag{5.12}$$

Expressions (5.11) and (5.12) can be used to study the artificial damping present in the approximation operator, even in the absence of structural damping, and its effect on signal attenuation and dispersion. Consider the solution of the one-degree-of-freedom problem previously described. Let $t = 0{\cdot}4$, $K = M = 1$, $C = 0$; then $\xi = 0{\cdot}16$ and $\eta = 0$. As a result,

$$R \doteq 0{\cdot}997;\ \theta \doteq 0{\cdot}202. \tag{5.13}$$

Since the modulus is supposed to be unity and the angle increment per time step is

$$\tfrac{1}{2}\omega\Delta t = 0{\cdot}2, \tag{5.14}$$

the numerical solution appears to exhibit only slight attenuation and dispersion in each time step. The cumulative effects are more striking, however. After one hundred time steps the attenuation can be estimated from the relation

$$\begin{aligned} R^{100} &\doteq (0{\cdot}997)^{100} = (1-0{\cdot}003)^{100} \\ &\doteq 1-(100)(0{\cdot}003) = 0{\cdot}7, \end{aligned} \tag{5.15}$$

indicating a substantial amount of cumulative artificial damping. Figure 9 can be used to visually verify (5.15).

One possible way to circumvent the artificial damping is to introduce compensatory damping analogous to the derivation in [18]. Returning to (5.11), the modulus can be unity only if

$$\eta = -\tfrac{1}{12}\xi \quad \text{or} \quad \eta = -3-\tfrac{1}{4}\xi. \tag{5.16}$$

This implies that, in the absence of structural damping in a one-degree-of-freedom system, there should be damping equivalent to (choosing the smaller negative root)

$$C = -\tfrac{1}{12}(\Delta t)K. \tag{5.17}$$

The results of applying this damping to the test problem can be seen in Figs. 9–11; the numerical solutions with and without compensatory damping are compared to the exact solution.

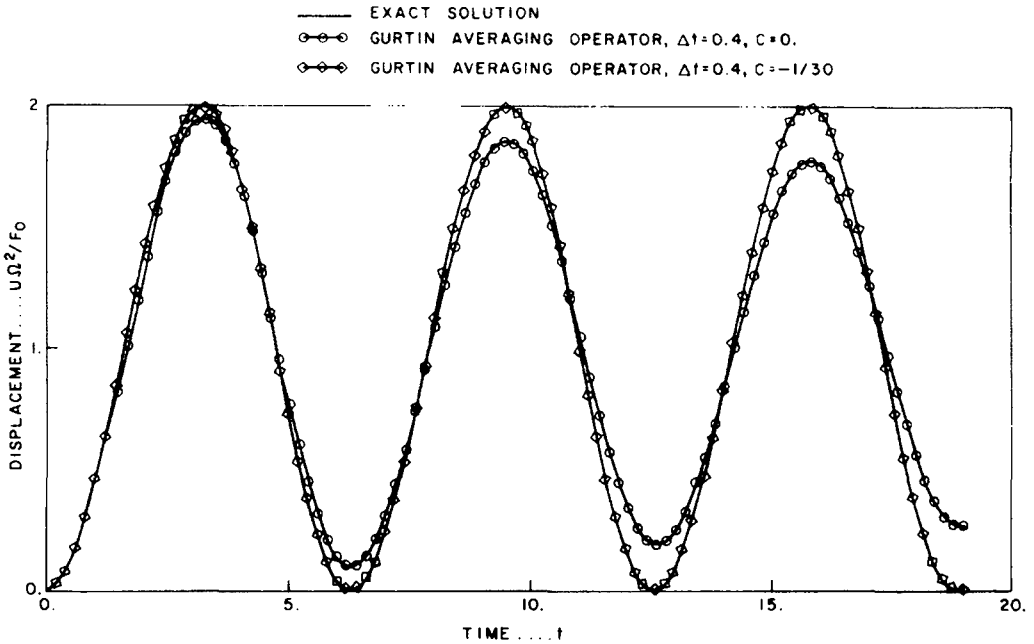

FIG. 9. Comparison of Gurtin averaging operator (with and without compensatory damping) to exact solution for $\Delta t = 0{\cdot}2$ (0·4).

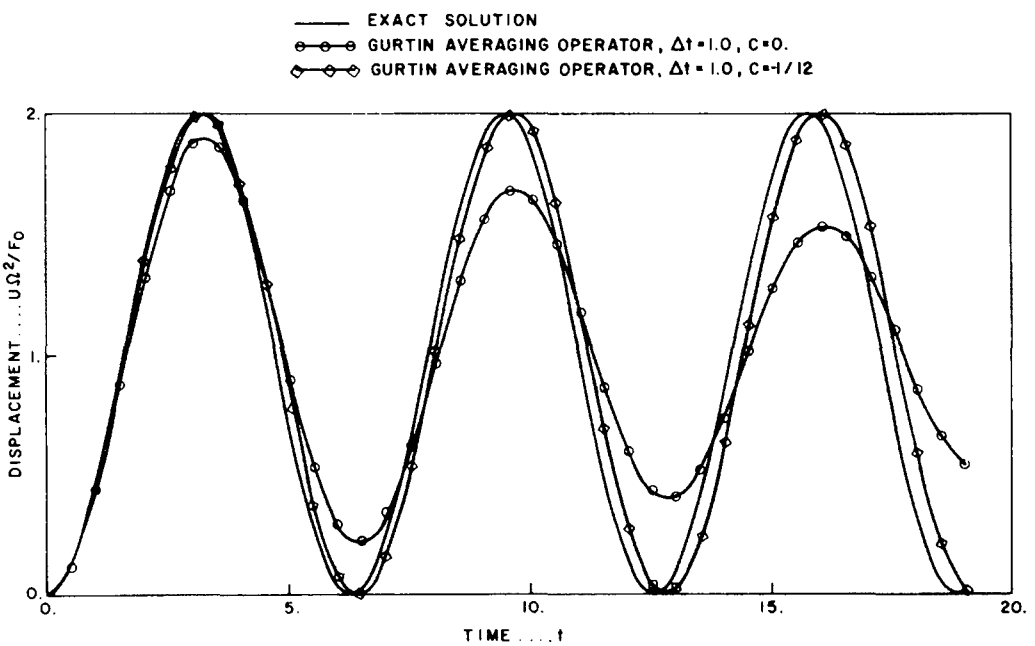

FIG. 10. Comparison of Gurtin averaging operator (with and without compensatory damping) to exact solution for $\Delta t = 0{\cdot}5$ (1·0).

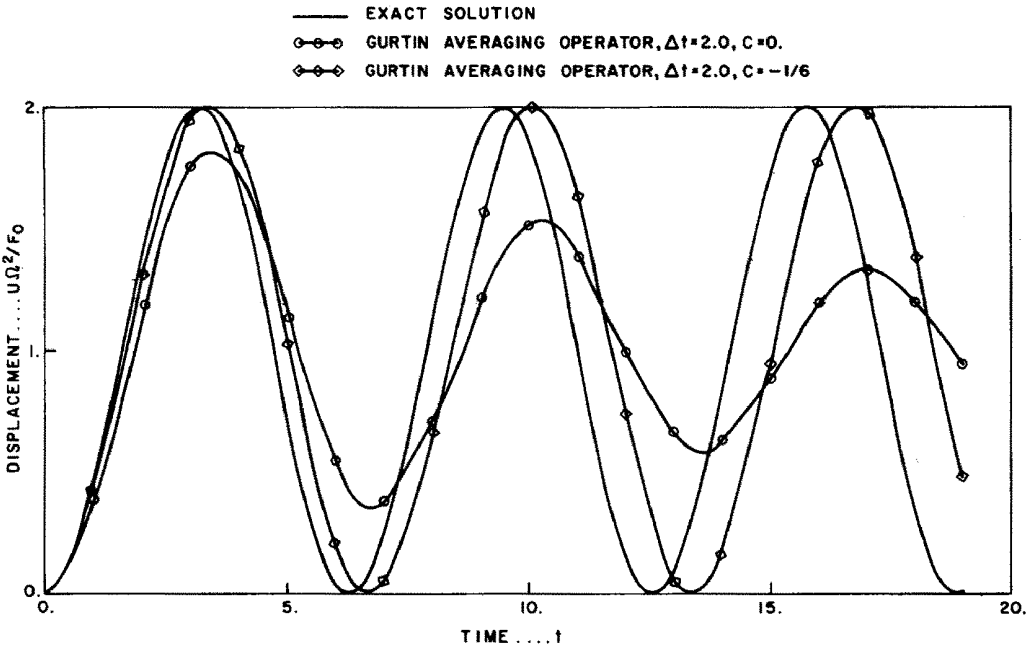

FIG. 11. Comparison of Gurtin averaging operator (with and without compensatory damping) to exact solution for $\Delta t = 1{\cdot}0$ (2·0).

The stability of the Gurtin averaging operator can be investigated easily for multi-degree-of-freedom systems by using the von Neumann procedure. First, the governing equation (5.8) is written in an equivalent form which eliminates the velocities and introduces the displacements at time t_{n-1}. Then

$$\left[[M]+\frac{\Delta t}{3}[C]+\frac{(\Delta t)^2}{12}[K]\right]\{u(t_a)\} = \frac{(\Delta t)^2}{24}\{2\{F(t_a)\}+3\{F(t_n)\}+\{F(t_{n-1}\}\}$$
$$+2\left[[M]+\frac{\Delta t}{12}[C]-\frac{(\Delta t)^2}{16}[K]\right]\{u(t_n)\}$$
$$-\left[[M]-\frac{\Delta t}{6}[C]+\frac{(\Delta t)^2}{24}[K]\right]\{u(t_{n-1})\}. \tag{5.18}$$

Following the same steps as those used in the stability calculations for the Wilson averaging operator, the characteristic equation for the error becomes

$$([M]^{-1}[K]-\omega^2[I])\{d\} = 0, \tag{5.19}$$

where

$$\omega^2 = -\frac{12(\lambda-1)^2}{(\Delta t)^2(\lambda+1)(\lambda+\frac{1}{2})}. \tag{5.20}$$

Since ω represents the natural frequencies of the system, eigenvalue pairs for the approximation operator which correspond to these frequencies are found to be:

$$\lambda_{1,2} = \frac{1 - \frac{1}{16}(\Delta t)^2\omega^2 \pm \frac{1}{12}i[36(\Delta t)^2\omega^2 - \frac{1}{16}(\Delta t)^4\omega^4]^{\frac{1}{2}}}{1 + \frac{1}{12}(\Delta t)^2\omega^2}. \tag{5.21}$$

Note the agreement between (5.9) and (5.21). This result implies that the Gurtin averaging method is unconditionally stable, since the selection of an arbitrarily large time step size does not yield an unbounded spectral radius for the approximation operator.

6. CONCLUSIONS

The stability of three widely used temporal approximation operators—the Newmark generalized acceleration method, the Wilson averaging method and the Gurtin averaging method—has been investigated. These three methods and two other procedures which have previously been examined—the central difference method [10] and the Houbolt method [9]—constitute the bulk of the algorithms being used today for the computation of structural dynamic response. Four of these methods are found to be unconditionally stable for all values of time step size: (1) the Houbolt backward difference formula; (2) the constant average acceleration version of the Newmark method; (3) the Wilson averaging method; and (4) the Gurtin averaging method. An added feature of the constant average acceleration method is that this operator contains no artificial attenuation, although some vibratory period error in the numerical solution occurs. Approximation operators (3) and (4) above contain both artificial attenuation and period error which are functions of the time step size and the natural structural frequencies of the system. A procedure to eliminate the artificial attenuation of the Gurtin averaging method by introducing negative damping, at least on single-degree-of-freedom systems, has been detailed.

Finally, it should be noted that, for modally uncoupled forced response calculations, there is little to choose between the various procedures; an explicit form, such as the central difference formula, whose stability can be controlled by an appropriate choice of time step size, is probably more economical and as accurate as any of the implicit formulas given here. When direct integration of the equations of motion is called for, however, a conservative estimate of the highest natural structural frequency of the system under study is required in order to assure the stability of the numerical solution; in this case, the time step limitation for a conditionally stable integration scheme may be prohibitive. A more reasonable procedure might be to use an unconditionally stable implicit scheme, such as one of the four mentioned previously, recognizing that artificial damping may distort the higher frequency components of the response (over-damping these modes in many instances). A proper understanding of the effects of artificial damping from the approximation operator should enable such results to be interpreted meaningfully.

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APPENDIX

For the Newmark generalized acceleration method:

$$\mathscr{K}_1 = \begin{bmatrix} K & C & M \\ 0 & 1 & -\gamma\Delta t \\ 1 & 0 & -\beta(\Delta t)^2 \end{bmatrix},$$

$$\mathscr{K}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \Delta t(1-\gamma) \\ 1 & \Delta t & (\Delta t)^2(\tfrac{1}{2}-\beta) \end{bmatrix},$$

$$\mathbf{u}(t_{n+1}) = \begin{Bmatrix} u(t_{n+1}) \\ \dot{u}(t_{n+1}) \\ \ddot{u}(t_{n+1}) \end{Bmatrix},$$

$$\mathbf{u}(t_n) = \begin{Bmatrix} u(t_n) \\ \dot{u}(t_n) \\ \ddot{u}(t_n) \end{Bmatrix},$$

$$F = \begin{Bmatrix} F(t_{n+1}) \\ 0 \\ 0 \end{Bmatrix},$$

$$\mathscr{A} = \frac{1}{1+\gamma\eta+\beta\xi}\begin{bmatrix} 1+\gamma\eta & \Delta t[1+\eta(\gamma-\beta)] & (\Delta t)^2\left[\left(\frac{1}{2}-\beta\right)+\eta\left(\frac{\gamma^2}{2}-\beta\right)\right] \\ -\frac{\gamma\xi}{\Delta t} & 1+\xi(\beta-\gamma) & \Delta t\left[(1-\gamma)+\xi\left(\beta-\frac{\gamma}{2}\right)\right] \\ -\frac{\xi}{(\Delta t)^2} & -\frac{(\eta+\xi)}{\Delta t} & -[\eta(1-\xi)+\xi(\tfrac{1}{2}-\beta)] \end{bmatrix},$$

and

$$\mathbf{G} = \frac{1}{1+\gamma\eta+\beta\xi}\begin{Bmatrix} \beta(\Delta t)^2 \\ \gamma\Delta t \\ 1 \end{Bmatrix}\frac{F(t_{n+1})}{M},$$

where $\eta = C\Delta t/M$ and $\xi = [(\Delta t)^2K]/M$. The quantities K, C and M are the stiffness, damping and mass of a one-degree-of-freedom system.

For the Wilson averaging operator:

$$\mathscr{K}_1 = \left[\begin{array}{ccc:ccc} K & C & M & 0 & 0 & 0 \\ 0 & -\frac{2}{\Delta t} & 1 & 0 & 0 & 0 \\ -\frac{3}{\Delta t} & 1 & 0 & 0 & 0 & 0 \\ \hdashline -\frac{1}{8} & 0 & 0 & 1 & 0 & 0 \\ -\frac{3}{4\Delta t} & 0 & 0 & 0 & 1 & 0 \\ -\frac{3}{(\Delta t)^2} & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

$$\mathscr{K}_0^T = \left[\begin{array}{ccc:ccc} 0 & 0 & -\frac{3}{\Delta t} & \frac{7}{8} & -\frac{3}{4\Delta t} & -\frac{3}{(\Delta t)^2} \\ 0 & -\frac{2}{\Delta t} & -2 & \frac{3\Delta t}{8} & \frac{1}{4} & -\frac{3}{\Delta t} \\ 0 & -1 & -\frac{\Delta t}{2} & \frac{(\Delta t)^2}{16} & \frac{\Delta t}{8} & -\frac{1}{2} \end{array}\right],$$

$$\mathbf{u}^T(t_{n+1}) = \langle u_{n+1}, \dot{u}_{n+1}, \ddot{u}_{n+1} \,\vdots\, u_m, \dot{u}_m, \ddot{u}_m\rangle,$$

$$\mathbf{u}^T(t_n) = \langle u_n, \dot{u}_n, \ddot{u}_n\rangle,$$

and

$$\mathscr{A} = \mathscr{K}_{0(2)} - \mathscr{K}_{1(21)}\mathscr{K}^{-1}_{1(11)}\mathscr{K}_{0(1)}.$$

For the original Gurtin operator:

$$\mathscr{K}_1 = \begin{bmatrix} \left[M+\frac{(\Delta t)^2}{12}K\right] & 0 & 0 \\ -2 & \Delta t & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathscr{K}_0 = \begin{bmatrix} \left[M-\frac{5}{12}(\Delta t)^2 K\right] & (\Delta t)\left[M-\frac{(\Delta t)^2}{12}K\right] & 0 \\ -2 & -\Delta t & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$\mathscr{A} = \frac{1}{1+\frac{1}{12}\xi}\begin{bmatrix} \left(1-\frac{5}{12}\xi\right) & \Delta t\left(1-\frac{1}{12}\xi\right) & 0 \\ -\frac{\xi}{\Delta t} & (1-\tfrac{1}{4}\xi) & 0 \\ 0 & 0 & (1+\tfrac{1}{12}\xi) \end{bmatrix}.$$

For the Gurtin averaging operator:

$$\mathscr{K}_1 = \begin{bmatrix} (1+\tfrac{1}{3}\eta+\tfrac{1}{12}\xi) & 0 & 0 \\ -4 & \Delta t & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathscr{K}_0 = \begin{bmatrix} (1+\tfrac{1}{3}\eta-\tfrac{1}{24}\xi) & \tfrac{1}{2}(\Delta t)(1+\tfrac{1}{12}\eta) & 0 \\ -4 & -\Delta t & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$\mathscr{A} = \frac{1}{(1+\frac{1}{3}\eta\times\frac{1}{12}\xi)}\begin{bmatrix} (1+\tfrac{1}{3}\eta-\tfrac{1}{24}\xi) & \tfrac{1}{2}(\Delta t)(1-\tfrac{1}{12}\eta) & 0 \\ -\frac{\xi}{2\Delta t} & (1-\tfrac{1}{6}\eta-\tfrac{1}{12}\xi) & 0 \\ 0 & 0 & (1+\tfrac{1}{3}\eta+\tfrac{1}{12}\xi) \end{bmatrix}.$$

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Абстракт—С целью определения устойчивости и приближенной вязкости, исследуются три непосредственные схемы интегрирования матричных уровнеиий динамики сооружений.

А именно: обобщенный оператор ускорения Ньюмарка, вариант усреднения линейного оператора ускорения Вильсона и метод усреднения, основанный на вариационном принципе, предложенным Гюртином. Используя методы решения, предложенные Й. Фон Нейманом и Рихтмеером оказывается, что последние два приближенные операторы неусловно стабольны. В добавлении находится также, что модификация постоянного ускорения для метода Ньюмарка неусловно стабильна и не затухает вследствие приближенной вязкости. Погрешность отбрасывания вследствие характеристик пропускания низких частот для пространственно дискретных систем оказывается высоко демофированная операторами усреднения Вильсона и Гюртина. Все три операторы дают погрешность в период действия реакции в виде скочкообразной функции времени.